Questions

Q1.

In a geometric series the common ratio is r and sum to n terms is S_n

Given

$$S_\infty = \frac{8}{7} \times S_6$$

show that $r = \pm \frac{1}{\sqrt{k}}$, where k is an integer to be found.

(4)

(Total for question = 4 marks)

Q2.

A competitor is running a 20 kilometre race.

She runs each of the first 4 kilometres at a steady pace of 6 minutes per kilometre.
After the first 4 kilometres, she begins to slow down.

In order to estimate her finishing time, the time that she will take to complete each subsequent kilometre is modelled to be 5% greater than the time that she took to complete the previous kilometre.

Using the model,

(a) show that her time to run the first 6 kilometres is estimated to be 36 minutes 55 seconds,

(2)

(b) show that her estimated time, in minutes, to run the rth kilometre, for $5 \le r \le 20$, is

$$6 \times 1.05^{r-4}$$

(1)

(c) estimate the total time, in minutes and seconds, that she will take to complete the race.

(4)

(Total for question = 7 marks)

Q3.

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

A geometric series has common ratio r and first term a.

Given $r \neq 1$ and $a \neq 0$

(a) prove that

$$S_n = \frac{a(1-r^n)}{1-r}$$

(4)

Given also that S_{10} is four times S_5

(b) find the exact value of r.

(4)

(Total for question = 8 marks)

Q4.

Show that

$$\sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n \cos(180n)^\circ = \frac{9}{28}$$

(Total for question = 3 marks)

Mark Scheme

Q1.

Question	Scheme	Marks	AOs
	Attempts $S_\infty = \frac{8}{7} \times S_6 \Rightarrow \frac{a}{1-r} = \frac{8}{7} \times \frac{a(1-r^6)}{1-r}$	M1	2.1
	$\Rightarrow 1 = \frac{8}{7} \times (1-r^6)$	M1	2.1
	$\Rightarrow r^6 = \frac{1}{8} \Rightarrow r = ..$	M1	1.1b
	$\Rightarrow r = \pm\frac{1}{\sqrt{2}}$ (so $k = 2$)	A1	1.1b
			(4 marks)

Notes:

M1: Substitutes the correct formulae for S_∞ and S_6 into the given equation $S_\infty = \frac{8}{7} \times S_6$

M1: Proceeds to an equation just in r

M1: Solves using a correct method

A1: Proceeds to $r = \pm\frac{1}{\sqrt{2}}$ giving $k = 2$

Q2.

Part	Working or answer an examiner might expect to see	Mark	Notes
(a)	$24 + (6 \times 1.05) + (6 \times 1.05^2)$ minutes	M1	This mark is for a method to find the time taken for the competitor to run 6 km
	$= 96.915$ minutes $= 36$ minutes 55 seconds	A1	This mark is given for finding the total time as required
(b)	For example, 5th km $= 6 \times 1.05^1$ 6th km $= 6 \times 1.05^2$ 7th km $= 6 \times 1.05^3 \ldots$ rth km $= 6 \times 1.05^{r-4}$	B1	This mark is given for showing the time taken to run the rth km, as required
(c)	$24 + \sum_{r=5}^{20} 6 \times 1.05^{r-4}$	M1	This mark is given for showing the total time to run the race is the time taken for the first 4 km added to the time taken from 5th to 20th km
	$= 24 + 6.3 \times \frac{(1.05^{16} - 1)}{1.05 - 1}$	M1	This mark is given for using $s = a\left(\frac{1-r^n}{1-r}\right)$ where $a = 6 \times 1.05 = 6.3$, $r = 1.05$ and $n = 20 - 4 = 16$
	$= 24 + 149.04$	A1	This mark is given for a correct total time (represented decimally)
	$= 173$ minutes and 3 seconds	A1	This mark is given for finding a correct total time given in minutes in seconds

Q3.

Question	Scheme	Marks	AOs
(a)	$S_n = a + ar + ar^2 + \ldots\ldots + ar^{n-1}$	B1	1.2
	$rS_n = ar + ar^2 + ar^3 + \ldots\ldots + ar^n \Rightarrow S_n - rS_n = \ldots$	M1	2.1
	$S_n - rS_n = a - ar^n$	A1	1.1b

	$S_n(1-r)=a(1-r^n)\Rightarrow S_n=\dfrac{a(1-r^n)}{(1-r)}$ *	A1*	2.1
		(4)	
(b)	$\dfrac{a(1-r^{10})}{1-r}=4\times\dfrac{a(1-r^5)}{1-r}$ or $4\times\dfrac{a(1-r^{10})}{1-r}=\dfrac{a(1-r^5)}{1-r}$ Equation in r^{10} and r^5 (and possibly $1-r$)	M1	3.1a
	$1-r^{10}=4(1-r^5)$	A1	1.1b
	$r^{10}-4r^5+3=0\Rightarrow(r^5-1)(r^5-3)=0\Rightarrow r^5=...$ or e.g. $1-r^{10}=4(1-r^5)\Rightarrow(1-r^5)(1+r^5)=4(1-r^5)\Rightarrow r^5=...$	dM1	2.1
	$r=\sqrt[5]{3}$ oe only	A1	1.1b
		(4)	
			(8 marks)

Notes:

(a)

B1: Writes out the sum or lists terms. Condone the omission of S.
The sum must include the first and last terms and (at least) two other correct terms and no incorrect terms e.g. ar^n
Note that the sum may be seen embedded within their working.

M1: For the key step in attempting to multiply the first series by r and subtracting.

A1: $S_n-rS_n=a-ar^n$ either way around but condone one side being prematurely factorised (but not both) **following correct work but this could follow B0 if insufficient terms were shown.**

A1*: Depends on all previous marks. Proceeds to given result showing all steps including seeing both sides unfactorised at some point in their working.
Note: If terms are <u>listed</u> rather than <u>added</u> then allow the first 3 marks if otherwise correct but withhold the final mark.

(b)

M1: For the correct strategy of producing an equation in just r^{10} and r^5 (and possibly $(1-r)$) with the "4" on either side using the result from part (a) and makes progress to at least cancel through by a
Some candidates retain the "$1-r$" and start multiplying out e.g. $(1-r)(1-r^{10})$ and this mark is still available as long as there is progress in cancelling the "a".

A1: Correct equation with the a and the $1-r$ cancelled. Allow any correct equation in just r^5 and r^{10}

dM1: Depends on the first M. Solves as far as $r^5=...$ by solving a 3 term quadratic in r^5 by a valid method – see general guidance or by difference of 2 squares – see above

A1: $r=\sqrt[5]{3}$ oe only. The solution $r=1$ if found must be rejected here.

(b) Note: For candidates who use $S_5=4S_{10}$ expect to see:

$4\times\dfrac{a(1-r^{10})}{1-r}=\dfrac{a(1-r^5)}{1-r}\Rightarrow 4(1-r^{10})=(1-r^5)$ M1A0

$4r^{10}-r^5-3=0\Rightarrow(4r^5+3)(r^5-1)=0\Rightarrow r^5=...$ or $4(1-r^5)(1+r^5)=(1-r^5)\Rightarrow r^5=...$ dM1A0

<u>Example for (a)</u>

a.

$S_n=a+ar+ar^2+ar^3...ar^{n-2}+ar^{n-1}$

$rS_n=ar+ar^2+ar^3+ar^4...ar^{n-1}+ar^n$

$S_n-rS_n=a(1-r^n)$

$S_n(1-r)=a(1-r^n)$

This scores B1M1A1A0:

B1: Writes down the sum including first and last terms and at least 2 other correct terms and no incorrect terms

M1: Multiplies by r and subtracts

A1: Correct equation (we allow one side to be prematurely factorised)

A0: One side was prematurely factorised

Q4.

Question	Scheme	Marks	AOs
	$a=\left(\frac{3}{4}\right)^2$ or $a=\frac{9}{16}$ or $r=-\frac{3}{4}$	B1	2.2a
	$\sum_{n=2}^{\infty}\left(\frac{3}{4}\right)^n\cos(180n)^\circ=\frac{\frac{9}{16}}{1-\left(-\frac{3}{4}\right)}=\ldots$	M1	3.1a
	$=\frac{9}{28}$*	A1*	1.1b
		(3)	

	Alternative 1:		
	$\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n \cos(180n)^\circ = \frac{-\frac{3}{4}}{1-\left(-\frac{3}{4}\right)} = \ldots$ or $r = -\frac{3}{4}$	B1	2.2a
	$\sum_{n=2}^{\infty}\left(\frac{3}{4}\right)^n \cos(180n)^\circ = -\frac{3}{7} - \left(-\frac{3}{4}\right)$	M1	3.1a
	$= \frac{9}{28}$ *	A1*	1.1b
	Alternative 2:		
	$\sum_{n=2}^{\infty}\left(\frac{3}{4}\right)^n \cos(180n)^\circ = \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^4 - \ldots$	B1	2.2a
	$= \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^4 + \ldots - \left(\frac{3}{4}\right)^3 - \left(\frac{3}{4}\right)^5 - \ldots$ $\left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^4 + \ldots = \left(\frac{3}{4}\right)^2\left(\frac{1}{1-\left(\frac{3}{4}\right)^2}\right)$ or $-\left(\frac{3}{4}\right)^3 - \left(\frac{3}{4}\right)^5 - \ldots = -\left(\frac{3}{4}\right)^3\left(\frac{1}{1-\left(\frac{3}{4}\right)^2}\right)$ $\sum_{n=2}^{\infty}\left(\frac{3}{4}\right)^n \cos(180n)^\circ = \left(\frac{3}{4}\right)^2\left(\frac{1}{1-\left(\frac{3}{4}\right)^2}\right) - \left(\frac{3}{4}\right)^3\left(\frac{1}{1-\left(\frac{3}{4}\right)^2}\right)$	M1	3.1a
	$= \frac{9}{28}$ *	A1*	1.1b
	Alternative 3:		
	$\sum_{n=2}^{\infty}\left(\frac{3}{4}\right)^n \cos(180n)^\circ = S = \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^4 - \ldots$	B1	2.2a
	$= \left(\frac{3}{4}\right)^2\left(1 - \left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)^2 - \ldots\right) = \left(\frac{3}{4}\right)^2\left(\frac{1}{4} + S\right) \Rightarrow \frac{7}{16}S = \frac{9}{64} \Rightarrow S = \ldots$	M1	3.1a
	$= \frac{9}{28}$ *	A1*	1.1b
			(3 marks)

Notes
B1: Deduces the correct value of the **first** term or the common ratio. The correct first term can be seen as part of them writing down the sequence but must be the **first** term. M1: Recognises that the series is infinite geometric and applies the sum to infinity GP formula with $a = \frac{9}{16}$ and $r = \pm\frac{3}{4}$ A1*: Correct proof **Alternative 1:** B1: Deduces the correct value for the sum to infinity (starting at $n = 1$) or the common ratio M1: Calculates the required value by subtracting the first term from their sum to infinity A1*: Correct proof **Alternative 2:** B1: Deduces the correct value of the **first** term or the common ratio. M1: Splits the series into "odds" and "evens", attempts the sum of both parts and calculates the required value by adding both sums A1*: Correct proof **Alternative 3:** B1: Deduces the correct value of the **first** term M1: A complete method by taking out the first term, expresses the rhs in terms of the original sum and rearranges for "S" A1*: Correct proof